Gravitation and the Local Symmetry Group of Space-Time

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Abstract.
According to general relativity, the interaction of a matter field with gravitation requires the simultaneous introduction of a tetrad field, which is a field related to translations, and a spin connection, which is a field assuming values in the Lie algebra of the Lorentz group. These two fields, however, are not independent. By analyzing the constraint between them, it is concluded that the relevant *local* symmetry group behind general relativity is provided by the Lorentz group. Furthermore, it is shown that the minimal coupling prescription obtained from the Lorentz covariant derivative coincides exactly with the usual coupling prescription of general relativity. Instead of the tetrad, therefore, the spin connection is to be considered as the fundamental field representing gravitation.

Keywords: General Relativity, Spin Connection, Tetrad, Symmetry Groups

1. Introduction

The group of motions of Minkowski spacetime is the ten–parameter Poincaré group, the semi–direct product of the translation and the Lorentz groups. Denoting by $\{x^a\}$ $(a, b, c, \ldots = 1, 2, 3, 4)$ the cartesian coordinates of Minkowski space, and by

$$\eta_{ab} = \mathrm{diag}(1, -1, -1, -1) \tag{1}$$

its metric tensor, an infinitesimal translation of the spacetime coordinates is defined as

$$\delta_t x^a = -i\epsilon^c \, P_c \, x^a, \tag{2}$$

where ϵ^c are the translation parameters, and

$$P_c = -i\frac{\partial}{\partial x^c} \equiv -i\partial_c \tag{3}$$

are the translation generators. By using these generators, the transformation (2) can be rewritten in the form

$$\delta_t x^a = -\epsilon^a. \tag{4}$$

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On the other hand, an infinitesimal Lorentz transformation is defined as

$$\delta_L x^a = -\frac{i}{2}\,\epsilon^{cd}\,L_{cd}\,x^a, \tag{5}$$

where $\epsilon^{cd} = -\epsilon^{dc}$ are the Lorentz parameters, and

$$L_{cd} = i(x_c\partial_d - x_d\partial_c) \tag{6}$$

are the Lorentz generators. By using these generators, the transformation (5) can be rewritten in the form

$$\delta_L x^a = -\epsilon^a{}_d\,x^d. \tag{7}$$

An interesting property of the Lorentz transformation (5) is that it can be rewritten formally as a translation (Kibble, 1961). In fact, by using the explicit form of L_{cd}, it becomes

$$\delta_L x^a = -i\,\xi^c\,P_c\,x^a, \tag{8}$$

which is a translation with

$$\xi^c = \epsilon^c{}_d\,x^d \tag{9}$$

as the translation parameters. In other words, an infinitesimal Lorentz transformation of the spacetime coordinates is equivalent to a translation with $\xi^c \equiv \delta_L x^c$ as the parameter. Actually, this is a property of the Lorentz generators L_{ab}, whose action can always be reinterpreted as a translation. The reason for such equivalence is that, because the Minkowski spacetime is transitive under translations, every two points related by a Lorentz transformation can also be related by a translation. Notice that the reverse is not true.

2. Conserved Quantities

The conservation laws of energy–momentum and angular–momentum in special relativity are connected with the Poincaré group, the isometry group of Minkowski spacetime (Trautman, 1962). In fact, according to Noether's theorem (Konopleva and Popov, 1981), the invariance of a physical system under a spacetime translation leads to the conservation of the *canonical* energy–momentum tensor, whereas the invariance under a Lorentz transformation leads to the conservation of the *canonical* angular–momentum tensor. When passing to general relativity, these two tensors are modified by the presence of gravitation. Furthermore, the source of the gravitational field, the so called *dynamical*

energy–momentum tensor, turns out to be a symmetrized version of the modified energy–momentum tensor.

Let us consider the following structure. At each point of spacetime, whose coordinates we denote by x^μ $(\mu, \nu, \rho, \ldots = 0, 1, 2, 3)$, we attach a Minkowski tangent space where both the Lorentz and the translation groups act locally. It should be remarked that the action of these two groups are not defined in a curved riemannian spacetime (Wald, 1984). Now according to the gauge approach to gravitation (Hehl et al, 1995), the gauge field related to translations shows up as the non–trivial part of the tetrad field (Kibble, 1961). Denoting by

$$B = B^a{}_\mu \, P_a \, dx^\mu \tag{10}$$

the translational gauge potential, which is a connection assuming values in the Lie algebra of the translation group, the tetrad field is written as (de Andrade and Pereira, 1997)

$$h^a{}_\mu = \partial_\mu x^a + c^{-2} B^a{}_\mu, \tag{11}$$

where the velocity of light c was introduced for dimensional reasons. Its inverse, denoted by $h^\rho{}_c$, is defined by the relations

$$h^a{}_\mu \, h^\mu{}_c = \delta^a{}_b \quad \text{and} \quad h^\mu{}_c \, h^c{}_\rho = \delta^\mu{}_\rho,$$

and is given by an infinite series:

$$h^\rho{}_c = \partial_c x^\rho - c^{-2} B^\rho{}_c + \ldots \; . \tag{12}$$

On the other hand, the gauge field related to Lorentz transformations is the so called spin connection $A^a{}_{b\mu}$, a connection assuming values in the Lie algebra of the Lorentz group. Its explicit form is (Dirac, 1958)

$$A^a{}_{b\mu} = h^a{}_\rho \, (\partial_\mu h^\rho{}_b + \Gamma^\rho{}_{\nu\mu} \, h^\nu{}_b) \equiv h^a{}_\rho \nabla_\mu h^\rho{}_b, \tag{13}$$

where $\Gamma^\rho{}_{\nu\mu}$ is the Levi–Civita connection of the spacetime metric $g_{\mu\nu}$, with ∇_μ the corresponding covariant derivative. The spacetime and the tangent space metrics are related by

$$g_{\mu\nu} = h^a{}_\mu \, h^b{}_\nu \, \eta_{ab}. \tag{14}$$

Let us consider now a general matter field Ψ with the action functional

$$S = \frac{1}{c} \int \mathcal{L} \, d^4x \equiv \frac{1}{c} \int L \, \sqrt{-g} \, d^4x, \tag{15}$$

where $g = \det(g_{\mu\nu})$. According to Noether's theorem, the *dynamical* energy–momentum tensor of the matter field — that is, the tensor

appearing in the right–hand side of the gravitational field equations — is given by

$$\mathcal{T}^{\mu}{}_{a} = -\frac{c^2}{h}\,\frac{\delta \mathcal{L}}{\delta B^{a}{}_{\mu}}, \tag{16}$$

where $h = \det(h^{a}{}_{\mu}) = \sqrt{-g}$. Since the tetrad is linear in the translational gauge field $B^{a}{}_{\mu}$, the functional derivative in relation to $B^{a}{}_{\mu}$ can alternatively be written as a functional derivative in relation to $h^{a}{}_{\mu}$,

$$\mathcal{T}^{\mu}{}_{a} = -\frac{1}{h}\,\frac{\delta \mathcal{L}}{\delta h^{a}{}_{\mu}}, \tag{17}$$

which is the form it usually appears in the literature (Weinberg, 1972). On the other hand, the angular–momentum tensor of the matter field is

$$\mathcal{J}^{\mu}{}_{ab} = \frac{1}{h}\,\frac{\delta \mathcal{L}}{\delta A^{ab}{}_{\mu}}. \tag{18}$$

It is important to remark that, as the *dynamical* energy–momentum tensor (17) is automatically symmetric,

$$h^{a\lambda}\,\mathcal{T}^{\mu}{}_{a} = h^{a\mu}\,\mathcal{T}^{\lambda}{}_{a}, \tag{19}$$

the *total* — that is, orbital plus spin — angular momentum tensor is given by (Weinberg, 1972; Hayashi, 1972)

$$\mathcal{J}^{\mu}{}_{ab} = x_a\,\mathcal{T}^{\mu}{}_{b} - x_b\,\mathcal{T}^{\mu}{}_{a}. \tag{20}$$

We see in this way that $\mathcal{T}^{\mu}{}_{a}$ and $\mathcal{J}^{\mu}{}_{ab}$ are not independent tensors. In fact, given the energy–momentum tensor, the expression for the angular–momentum tensor can immediately be written down.

That $\mathcal{T}^{\mu}{}_{a}$ and $\mathcal{J}^{\mu}{}_{ab}$ are not independent tensors should not be surprising because the translational gauge potential $B^{a}{}_{\mu}$ and the spin connection $A^{ab}{}_{\mu}$ are not independent either, as can be seen from Eq.(13), which gives the spin connection $A^{ab}{}_{\mu}$ in terms of the translational gauge potential $B^{a}{}_{\mu}$. The physical reason for this dependency is that both $B^{a}{}_{\mu}$ and $A^{ab}{}_{\mu}$ are produced by the very same gravitational field.

Let us then look for the inverse relation, that is, let us look for an expression yielding $B^{a}{}_{\mu}$ in terms of $A^{ab}{}_{\mu}$. By comparing the expressions (16) and (18) with (20), we find immediately that

$$B^{a}{}_{\mu} = c^2\,A^{a}{}_{b\mu}\,x^{b}. \tag{21}$$

In fact, from Eq. (18), and making use of Eq. (16), we have

$$\mathcal{J}^{\mu}{}_{ab} = -c^{-2}\,\mathcal{T}^{\rho}{}_{c}\,\frac{\delta B^{c}{}_{\rho}}{\delta A^{ab}{}_{\mu}}. \tag{22}$$

But, taking into account that $A^{ab}{}_{\mu} = -A^{ba}{}_{\mu}$, we get from (21)

$$\frac{\delta B^{c}{}_{\rho}}{\delta A^{ab}{}_{\mu}} = c^2\, \delta^{\mu}{}_{\rho} (\delta^{c}{}_{a}\, x_b - \delta^{c}{}_{b}\, x_a). \tag{23}$$

Substituting in (22), we get exactly the expression (20) for the angular momentum $\mathcal{J}^{\mu}{}_{ab}$.

3. Minimal Coupling Prescription

When considering coordinate transformations, only the generators P_a and L_{ab} must be taken into account. However, in the study of the coupling of a general matter field to gravitation, other representations of the Lorentz generators show up. For example, under a *local* tangent–space Lorentz transformation, a general matter field $\Psi(x^{\mu})$ will change according to (Ramond, 1989)

$$\delta\Psi \equiv \Psi'(x) - \Psi(x) = -\frac{i}{2}\, \epsilon^{ab} J_{ab} \Psi, \tag{24}$$

where J_{ab} is an appropriate generator of the infinitesimal Lorentz transformations. The most general form of J_{ab} is

$$J_{ab} = L_{ab} + S_{ab}, \tag{25}$$

where L_{ab} is the *orbital* part of the generator, whose explicit form, given by (6), is the same for all fields, and S_{ab} is the *spin* part of the generator, whose explicit form depends on the spin contents of the field Ψ. Notice that the orbital generators L_{ab} are able to act in the spacetime argument of $\Psi(x^{\mu})$ due to the relation

$$\partial_a = (\partial_a x^{\mu})\, \partial_{\mu}.$$

By using the explicit form of L_{ab}, the Lorentz transformation (24) can be rewritten as

$$\delta\Psi = -\epsilon^{ab}\, x_b\, \partial_a \Psi - \frac{i}{2}\, \epsilon^{ab} S_{ab} \Psi, \tag{26}$$

or equivalently,

$$\delta\Psi = -\xi^{c} \partial_c \Psi - \frac{i}{2}\, \epsilon^{ab} S_{ab} \Psi, \tag{27}$$

where use has been made of Eq. (9). In other words, the *orbital* part of the transformation can be reduced to a translation, and consequently the Lorentz transformation of a general field Ψ can be rewritten as a translation plus a strictly spin Lorentz transformation. It should be remarked that, despite the similarity with a Poincaré transformation, it

does not correspond to a transformation of the Poincaré group because in this group the translation and the Lorentz parameters are completely independent. This is clearly not the case here because of the constraint (9) between the translation and the Lorentz parameters.

As is well known, the gravitational minimal coupling prescription amounts to replace all flat–spacetime ordinary derivatives ∂_a by covariant derivatives $\mathcal{D}_a$. The general definition of covariant derivative is (Aldrovandi and Pereira, 1995)

$$\mathcal{D}_c \Psi = \partial_c \Psi + \frac{1}{2} A^{ab}{}_c \frac{\delta \Psi}{\delta \epsilon^{ab}}, \tag{28}$$

where $A^{ab}{}_c = A^{ab}{}_\mu h^\mu{}_c$. Substituting (26), we get

$$\mathcal{D}_c \Psi = \partial_c \Psi - A^{ab}{}_c \, x_b \, \partial_a \Psi - \frac{i}{2} A^{ab}{}_c \, S_{ab} \, \Psi, \tag{29}$$

or equivalently,

$$\mathcal{D}_c \Psi = (\delta^a{}_c - A^a{}_{bc} x^b) \partial_a \Psi - \frac{i}{2} A^{ab}{}_c \, S_{ab} \, \Psi. \tag{30}$$

Then, by making use of Eqs. (11) and (21), we can write

$$\mathcal{D}_c \Psi = h^\mu{}_c \, \mathcal{D}_\mu \Psi, \tag{31}$$

with

$$\mathcal{D}_\mu = \partial_\mu - \frac{i}{2} A^{ab}{}_\mu \, S_{ab} \tag{32}$$

the Fock–Ivanenko covariant derivative operator (Fock and Ivanenko, 1929; Fock, 1929). Therefore, the minimal coupling prescription associated with the transformation (26) can be stated in the form

$$\partial_c \to \mathcal{D}_c = h^\mu{}_c \, \mathcal{D}_\mu, \tag{33}$$

which is exactly the usual coupling prescription of general relativity. In fact, as is well known, in the coupling prescription of general relativity the tetrad $h^a{}_\mu$ and the spin connection $A^{ab}{}_\mu$ are not independent fields. Such a coupling prescription, as we have shown, can be obtained from a Lorentz covariant derivative with the complete representation (25). In this covariant derivative, the *orbital* part of the Lorentz generators are reduced to a translation, which gives rise to a tetrad that depends on the spin connection. This reduction, therefore, is the responsible for the constraint between the tetrad field and the spin connection. The same constraint gives rise also to the relation between energy–momentum and angular–momentum tensors of a matter field.

4. Final Remarks

The basic results of this paper can be summarized in the following way. As is well–known, the energy–momentum conservation is related to the invariance of the action under a translation of the spacetime coordinates, and the angular–momentum conservation is related to the invariance of the action under a Lorentz transformation. However, as the symmetric energy–momentum tensor $\mathcal{T}^{\mu}{}_{a}$ and the *total* angular–momentum tensor $\mathcal{J}^{\mu}{}_{ab}$ are not independent quantities, the parameters related to translation and Lorentz transformation can not be independent either. In fact, they are related by

$$\xi^a = \epsilon^a{}_b \, x^b, \tag{34}$$

which yields naturally the relation (20) between $\mathcal{T}^{\mu}{}_{a}$ and $\mathcal{J}^{\mu}{}_{ab}$.

On the other hand, we have shown that the minimal coupling prescription associated with the Lorentz transformation (26), that is, the coupling prescription given by a derivative covariant under the Lorentz transformation (26), yields exactly the coupling prescription of general relativity, provided the identification

$$A^a{}_{b\mu} \, x^b = c^{-2} \, B^a{}_{\mu} \tag{35}$$

be made. This identification implies that the tetrad field and the spin connection are not independent fields. As a consequence, the *local* symmetry group of general relativity can not be the Poincaré group because in this group there are *ten* independent parameters ϵ^a and ϵ^{ab}, and ten independent gauge fields $B^a{}_{\mu}$ and $A^a{}_{b\mu}$. The true *local* symmetry group behind general relativity, therefore, is the six–parameter Lorentz group. In the form (27), the Lorentz transformation of a matter field resembles a Poincaré transformation, but due to the *four* constraints (34), it is actually a transformation of the Lorentz group. In fact, if the local symmetry group were given by the Poincaré group, the tetrad and the spin connection would be independent fields.

We have also seen that the tetrad field appears naturally in the theory as a consequence of the reduction of the *orbital* Lorentz generator L_{ab} to a translation in the coupling prescription. The resulting tetrad field,

$$h^a{}_{\mu} = \partial_\mu x^a + A^a{}_{b\mu} \, x^b, \tag{36}$$

is a functional of the spin connection, which reduces to the usual form (11) when the identification (35) is used. In agreement with the fact that the local symmetry group of general relativity is the Lorentz group, therefore, we can then say that the fundamental field of gravitation is the spin connection and not the tetrad.

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